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COMMENT

Comment on 'Singular point analysis, resonances and Yoshida's theorem'

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Abstract. We show that the paper by Steeb *et al*, criticising Yoshida's theorem, is based on a misunderstanding of the latter.

In a recent publication, Steeb *et al* [1] have presented criticisms of a theorem by Yoshida [2] which has related the existence of algebraic integrals for weighted homogeneous systems of ordinary differential equations (ODE) with the singular structure of their solutions. The criticism of Steeb *et al* has its origins in the fact that they believe that Yoshida's theorem assumes that the scaling properties determine the dominant singular behaviour of the system. However, such an assumption is neither needed nor present in the theorem. In fact, the misunderstanding of the authors appears clearly in the second paragraph of the article where the Kowalevski exponents, introduced in Yoshida's work, are confused with the resonances, introduced in the Painlevé method by Ablowitz *et al* ([3], hereafter referred to as ARS). The distinction is admittedly a subtle one. We have discussed the matter at length in [4]. The difference between the two objects makes the criticisms of Steeb *et al* unfounded and ensure the validity of Yoshida's theorem. In order to make the present comment as self-contained as possible we present the correct formulation of Yoshida's theorem together with an illustrative example and then comment briefly on the examples analysed by Steeb *et al.*

We start from a system of first-order autonomous ODE

$$dx_i/dt = F_i(x_1, \ldots, x_n) \tag{1}$$

which we assume invariant under a scaling transformation:

$$t \to \varepsilon^{-1} t,\ x_i \to \varepsilon^{g_i} x_i. \tag{2}$$

(One basic assumption of the theorem is that the F_i are rational functions and that the weights g_i are rational numbers.) In this case there exist particular solutions of the system of the form

$$x_i = a_i t^{-g_i} \tag{3}$$

(the 'scaling' solutions). The constants a_i are obtained from the solution of the algebraic equations:

$$F_i(a_1, \ldots, a_n) = -g_i a_i. \tag{4}$$

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Next we consider the variational equations around the solution (3). Writing

$$x_i = (a_i + \xi_i)t^{-g_i} \tag{5}$$

we obtain the linearised equations

$$t\,\mathrm{d}\xi_i/\mathrm{d}t = \sum_j K_{ij}\xi_j \tag{6}$$

where the matrix K_{ij} is given by $K_{ij} = \partial F_i/\partial x_j|_{x=a} + \delta_{ij}g_j$. The eigenvalues of K are called the Kowalevski exponents. Now the Yoshida theorem, referred to in [1], states that for system (1) to be algebraically integrable it is necessary that all Kowalevski exponents be rational numbers. Moreover, each Kowalevski exponent can be associated to the weight of a constant of motion. Note that in the theorem it is never assumed that all a_i are non-vanishing. Some of them may indeed vanish; the minimum needed is that at least one $a_j g_j \neq 0$. Still, the Kowalevski exponents are measured away from the leading behaviour $a_i t^{-g_i}$ independently on the vanishing of the coefficient a_i. Let us illustrate this with a very simple example. Consider the scale-invariant system

$$\mathrm{d}x/\mathrm{d}t = -x^2 \qquad \mathrm{d}y/\mathrm{d}t = xy + y^2. \tag{7}$$

Two different singular behaviours, in the Painlevé sense, can be found ($\tau = t - t_0$):

$$x = 1/\tau \qquad y = -2/\tau \tag{8}$$

with resonances (obtained using the ARS algorithm) $r = -1, -2$ and a second one:

$$x = 1/\tau \qquad y = \alpha\tau \qquad \text{with } \alpha \text{ free} \tag{9}$$

with resonances $r = -1$ and 0. Clearly the first case corresponds to both a_i non-vanishing while the second case of singularities occurs when $a_2 = 0$, i.e. the coefficient of $1/\tau$ in y vanishes. Now the ARS resonance $r = 0$ indicates that the coefficient α of the term $\alpha\tau$ is a free constant. However, the Kowalevski exponents are not measured with the actual singular behaviour for the starting point but with the scaling one, i.e. $1/\tau$. We should write

$$y = \tau^{-1}(0 + \alpha\tau^2).$$

Therefore one finds for the Kowalevski exponents -1 and $+2$. The latter indicates that the free constant enters two orders after the scaling behaviour which happens to have a vanishing coefficient. The system can, of course, be completely integrated. One finds

$$x = 1/\tau \qquad y = 2c\tau/(1 - c\tau^2) \tag{10}$$

where c is the second integration constant, which we can express in terms of x and y as

$$c = x^2 y/(y + 2x). \tag{11}$$

As Yoshida's theorem states, the Kowalevski exponent is indeed related to the weight of the integral, which is this case is equal to 2. One can also check that the gradient of the integral computed on the scaling singular solution is non-vanishing. Thus Yoshida's theorem is fully satisfied and one sees that the resonances do not coincide with the Kowalevski exponents [4]. Thus one should not use them indiscriminately, and only the Kowalevski exponents are directly related to the degree of the integrals of motion.

In the example above, the scalings of both x and y were unique. Steeb *et al* [1] presented examples with one or two arbitrary scalings introduced through the proper bilinearities on the ODE considered. Free scalings, however, do not change anything either in the approach or in its conclusions. The first system considered was

$$dx/dt = xy \qquad dy/dt = -xy \qquad dz/dt = z(x-y) \tag{12}$$

which is scale invariant under the scaling

$$t \to \varepsilon^{-1}t \qquad x \to \varepsilon x \qquad y \to \varepsilon y \qquad z \to \varepsilon^{\alpha} z$$

with α free. The constants of the motion are $I_1 = x+y$ and $I_2 = xyz$ with respective weights 1 and $2+\alpha$ (and non-zero gradients). Steeb *et al* have (correctly) calculated the resonances as being $-1, 0$ and 1, and have (incorrectly) deduced that Yoshida's theorem does not apply here. However, the Kowalevski exponents, computed as defined by Yoshida [2], being related to the precise scale invariance of the system, turn out to be -1, 1 and $2+\alpha$. These exponents, therefore, give precisely the order of the invariants. The difference between ARS resonances and Kowalevski exponents can be better understood when one looks at the leading singular behaviour:

$$x \sim \tau^{-1} \qquad y \sim \tau^{-1} \qquad z \sim \lambda\tau^{2} \qquad \text{with } \lambda \text{ free (hence the resonance 0)} \tag{13}$$

considered. The dependence of z can be rewritten:

$$z \sim \tau^{-\alpha}(0 + \lambda\tau^{2+\alpha})$$

which shows that the coefficient of the leading scaling behaviour vanishes but that the free constant λ enters correctly at the order $2+\alpha$ with respect to it.

Similar conclusions can be reached for the second example of Steeb *et al* [1]:

$$dx/dt = xy \qquad dy/dt = -xy \qquad dz/dt = -zy \qquad dw/dt = wx \tag{14}$$

with integrals $I_1 = x+y$, $I_2 = xyzw$, $I_3 = xz + zw$. As I_2, however, has zero gradient and I_3 is not, in general, weighted homogeneous for the scaling ($x \to \varepsilon x, y \to \varepsilon y$, $z \to \varepsilon^{\alpha} z$, $w \to \varepsilon^{\beta} w$) introduced by Steeb *et al*, one should rather consider $I_2' = xz$ and $I_3' = yw$ with respective weights $1+\alpha$ and $1+\beta$. A straighforward calculation shows that the Kowalevski exponents are -1, 1, $1+\alpha$ and $1+\beta$ in perfect agreement with Yoshida's theorem.

Thus in the light of our analysis it appears that the conclusion of Steeb *et al* is unfounded and results from a misunderstanding of the distinction between the Kowalevski exponents and the ARS resonances.

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